

Loop-erased random walk on the Sierpinski gasket

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Abstract

We consider a model of loop-erased random walks on the finite pre-Sierpiński gasket which permits rigorous analysis. We prove the existence of the scaling limit and show that the path of the limiting process is almost surely self-avoiding, while having Hausdorff dimension strictly greater than 1. This result means that the path has infinitely fine creases, while having no self-intersection. Our loop-erasing procedure is formulated by a ‘larger-scale-loops-first’ rule. It enables us to obtain exact recursion relations, making use of ‘self-similarity’ of a fractal structure.

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Key words: loop-erased random walk, scaling limit, fractal, Sierpinski gasket, displacement exponent

1 Introduction

In this paper, we consider a model of loop-erased random walks on the finite pre-Sierpiński gasket which permits rigorous analysis.

A loop-erased random walk is a kind of self-avoiding walk, which is a random walk that cannot visit any point more than once. Concerning self-avoiding walks, there have been questions that are simple to ask but difficult to answer, such as: How far can an n -step self-avoiding walk go in average? Does it have a scaling-limit? The non-Markov property of the walk makes the matter so difficult that we still do not know rigorous proofs for the ‘standard’ model on the low-dimensional (2- and 3- dimensional) square lattices, which corresponds to the uniform measure on self-avoiding paths of a given length ([14]). As such, we believe a self-avoiding walk on the pre-Sierpiński gasket (a lattice version) serves as an interesting low-dimensional model, since it is solvable.

In [6, 7, 9, 5], models for self-avoiding walks on the 2- and 3-dimensional pre-Sierpiński gasket were investigated, and a positive answer to the second question, above, was established; in addition, some path properties of the limit process were proved such as Hausdorff dimensions, Hölder continuity, whether the limit is also self-avoiding, and so on. In [8, 9], some results were provided with regard to the first question. The values of the mean-square displacement exponents obtained earlier by scaling arguments in physics literature were proved.

On the other hand, Lawler [12] defined a loop-erased random walk on square lattices, which is a process obtained by chronologically erasing the loops from a simple random walk. It is another kind of self-avoiding walk, but in this case, one can make use of the properties of simple random walks, on which there has been much study, for analysis. The scaling limit of the loop-erased random

walk on the 2-dimensional lattice has been studied, using Schramm Loewner Evolution (SLE). To name a few works in this line, [13], [15]. In [11], Kozma proved the existence of the scaling limit of the 3-dimensional loop-erased random walk.

In this paper, we define a loop-erased random walk on the pre-Sierpiński gasket by employing a ‘larger-scale-loops-first’ rule, which enables us to obtain recursion relations, making use of ‘self-similarity’ of a fractal structure, instead of translational invariance of the square lattices. Our loop-erased walk will also be self-avoiding, but we shall show that it belongs to a different universality class from the self-avoiding walk with uniform measure. We shall also prove the existence of the scaling limit, and that the path of the limiting process is almost surely self-avoiding, while having Hausdorff dimension $\log\{\frac{1}{15}(20 + \sqrt{205})\}/\log 2 = 1.1939 \dots$. This result means that the path has infinitely fine creases, while having no self-intersection.

Shinoda [16] obtained the exponent for the mean-square displacement for loop-erased random walks on the pre-Sierpiński gasket through uniform spanning trees. In the physics literature, D. Dhar and A. Dhar [3] investigated the distribution of sizes of erased loops in terms of spanning tree and scaling arguments. Our path Hausdorff dimension is consistent with their results, so it is our belief that our larger-scale-loops-first formulation is a natural procedure to study.

In Section 2, we describe the set-up of our model and the loop-erasing procedure, and show that the asymptotics of path length is consistent with the results in [3] and [16]. Section 3 is devoted to the examination of scaling limit.

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2 Paths on the pre-Sierpiński gaskets

2.1 The pre-Sierpiński gaskets.

We consider the pre-Sierpiński gasket, a lattice version of the Sierpiński gasket, which is a fractal with Hausdorff dimension $\log 3/\log 2$. (For fractals, see [4].) Let us recall the definition of the pre-Sierpiński gasket: by denoting $O = (0, 0)$, $a_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b_0 = (1, 0)$, and for each $N \in \mathbb{N}$, $a_N = 2^N a_0$, $b_N = 2^N b_0$, then define F'_0 be the graph that consists of three vertices and three edges of $\triangle Oa_0b_0$ and define the recursive sequence of graphs $\{F'_N\}_{N=0}^\infty$ by

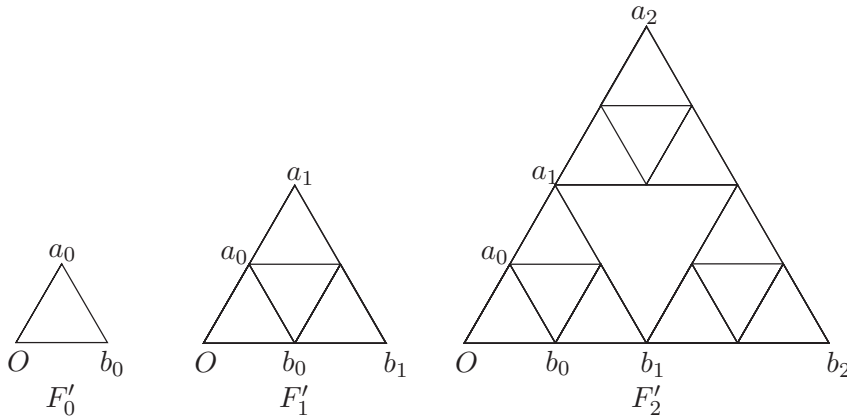


Fig. 1

$$F'_{N+1} = F'_N \cup (F'_N + a_N) \cup (F'_N + b_N), \quad N \in \mathbb{Z}_+ = \{0, 1, 2, \dots\},$$

where $A + a = \{x + a : x \in A\}$ and $kA = \{kx : x \in A\}$. F'_0 , F'_1 and F'_2 are shown in Fig. 1.

Finally, we let F''_N be the union of F'_N and its reflection with respect to the y -axis, and denote $F_0 = \bigcup_{N=1}^{\infty} F''_N$; the graph F_0 is called the (infinite) **pre-Sierpiński gasket**. F_0 is shown in Fig. 2.

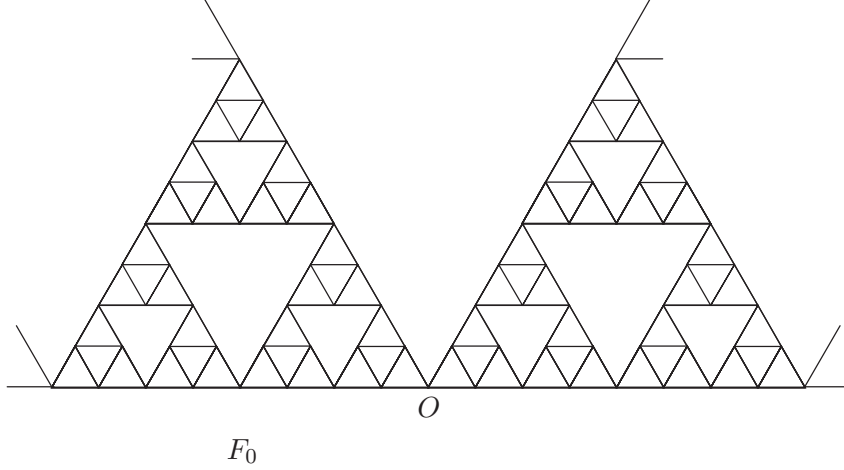


Fig. 2

Furthermore, by letting G_0 and E_0 denote the set of vertices and the set of edges of F_0 , respectively, we see that, for each $N \in \mathbb{Z}_+$, $F_N = 2^N F_0$ can be regarded as a coarse graph with vertices $G_N = \{2^N x : x \in G_0\}$ and edges $E_N = \{2^N \overline{xy} : \overline{xy} \in E_0\}$. Given $x \in G_N$, let $\mathcal{N}_N(x)$ be the four nearest neighbors of x on F_N , that is, $\mathcal{N}_N(x) = \{y \in G_N : \overline{xy} \in E_N\}$.

2.2 Paths on the pre-Sierpiński gaskets.

Let us denote the set of finite paths on F_0 by

$$W = \{ w = (w(0), w(1), \dots, w(n)) : w(0) \in G_0, w(i) \in \mathcal{N}_0(w(i-1)), 1 \leq i \leq n, n \in \mathbb{N} \},$$

and the set of finite paths on F_0 starting at O by

$$W^* = \{ w \in W : w(0) = O \}.$$

This gives the natural definition for the length ℓ of a path $w = (w(0), w(1), \dots, w(n)) \in W$; namely, $\ell(w) = n$.

For a path $w \in W$ and $A \subset G_0$, we define the hitting time of A by

$$T_A(w) = \inf\{j \geq 0 : w(j) \in A\},$$

where we set $\inf \emptyset = \infty$. By taking $w \in W$ and $M \in \mathbb{Z}_+$, we shall define the recursive sequence $\{T_i^M(w)\}_{i=0}^m$ of **hitting times of G_M** as follows: Let $T_0^M(w) = T_{G_M}$, and for $i \geq 1$, let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) : w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\};$$

here we take m to be the smallest integer such that $T_{m+1}^M(w) = \infty$. Then $T_i^M(w)$ can be interpreted as being the time taken for the path w to hit vertices in G_M for the $(i+1)$ -th time, under the condition that if w hits the same vertex in G_M more than once in a row, we count it only once.

Now we consider two sequences of subsets of W^* as follows: for each $N \in \mathbb{Z}_+$, let the set of paths from O to a_N , which do not hit any other vertices in G_N on the way, be

$$W_N = \{w = (w(0), w(1), \dots, w(n)) \in W^* : w(n) = a_N, n = T_1^N(w)\},$$

and let the set of paths from O to a_N that hit b_N ‘once’ on the way (subject to the counting rule explained above) be

$$V_N = \{w = (w(0), w(1), \dots, w(n)) \in W^* : w(n) = a_N, w(T_1^N(w)) = b_N, n = T_2^N(w)\}.$$

Then for a path $w \in W$ and $M \in \mathbb{Z}_+$, we define the **coarse-graining map** Q_M by

$$(Q_M w)(i) = w(T_i^M(w)), \quad \text{for } i = 0, 1, 2, \dots, m,$$

where m is the smallest integer such that $T_{m+1}^M(w) = \infty$. Thus,

$$Q_M w = [w(T_0^M(w)), w(T_1^M(w)), \dots, w(T_m^M(w))]$$

is a path on a coarser graph F_M . For $w \in W_N \cup V_N$ and $M \leq N$, the end point of the coarse-grained path is $w(T_m^M(w)) = a_N$, and if we write $(2^{-M} Q_M w)(i) = 2^{-M} w(T_i^M(w))$, then $2^{-M} Q_M w$ is a path in $W_{N-M} \cup V_{N-M}$ and $\ell(2^{-M} Q_M w) = m$. Notice that if $M \leq N$, then $Q_N \circ Q_M = Q_N$. Throughout the following, we write simply $w(T_i^M)$ instead of $w(T_i^M(w))$.

2.3 Loop-erased paths.

Let Γ be the set of self-avoiding paths starting at O :

$$\Gamma = \{ (w(0), w(1), \dots, w(n)) \in W^* : w(i) \neq w(j), i \neq j, n \in \mathbb{N} \},$$

and let us denote the following two subsets of Γ :

$$\hat{W}_N = W_N \cap \Gamma, \quad \hat{V}_N = V_N \cap \Gamma.$$

For $(w(0), w(1), \dots, w(n)) \in W^*$, We call a path segment $[w(i), w(i+1), \dots, w(j)]$ a loop if there are i, j , $0 \leq i < j \leq n$ such that $w(i) = w(j)$ and $w(k) \neq w(i)$ for any $i < k < j$.

We shall now describe a loop-erasing procedure for paths in $W_1 \cup V_1$:

- (i) Erase all the loops formed at O ;
- (ii) Progress one step forward along the path, and erase all the loops at the new position;
- (iii) Iterate this process, taking another step forward along the path and erasing the loops there, until reaching a_1 (the endpoint of all paths in W_1 and V_1).

To be precise, for $w \in W_1 \cup V_1$, define the recursive sequence $\{s_i\}_{i=0}^m$

$$s_0 = \sup\{j : w(j) = O\},$$

$$s_i = \sup\{j : w(j) = w(s_{i-1} + 1)\}.$$

If $s_i > s_{i-1} + 1$, then $[w(s_{i-1} + 1), w(s_{i-1} + 2), \dots, w(s_i - 1), w(s_i)]$ forms a loop, starting and ending at $w(s_{i-1} + 1) = w(s_i)$. We erase it by removing all of the points $w(s_{i-1} + 1), w(s_{i-1} + 2), \dots, w(s_i - 2), w(s_i - 1)$. If $w(s_m) = a_1$, then we have obtained a loop-erased path,

$$Lw = [w(s_0), w(s_1), \dots, w(s_m)] \in \hat{W}_1 \cup \hat{V}_1.$$

Note that $w \in W_1$ implies $Lw \in \hat{W}_1$, but that $w \in V_1$ can result in $Lw \in \hat{W}_1$, with b_1 being erased together with a loop. So far, our loop-erasing procedure is the same as that defined for paths on \mathbb{Z}^d in [12].

We shall generalize the above procedure to a loop-erasing procedure for a path w in $W_N \cup V_N$ that yields a self-avoiding path in $\hat{W}_N \cup \hat{V}_N$. The idea is to first erase loops of ‘largest scale’, and then go down to ‘smaller scales’ step by step. For this purpose, we need the notion of ‘skeletons’.

Let \mathcal{T}_M be the set of all upward (closed and filled) triangles which are translations of $\triangle Oa_Mb_M$ and whose vertices are in G_M ; an element of \mathcal{T}_M is called a 2^M -**triangle**. For $w \in W$ and $M \geq 0$, we shall define a sequence $(\Delta_1, \dots, \Delta_k)$ of 2^M -triangles w ‘passes through’ and a sequence $\{T_i^{ex,M}(w)\}_{i=1}^k$ of exit times from them as a subsequence of $\{T_i^M(w)\}_{i=1}^m$, as follows: We start by defining $T_0^{ex,M}(w) = T_0^M(w)$. (Thus If $w \in W^*$, then $T_0^{ex,M}(w) = 0$.) There is a unique element of \mathcal{T}_M that contains $w(T_0^M)$ and $w(T_1^M)$, which we denote by Δ_1 . For $i \geq 1$, define

$$j(i) = \min\{j \geq 0 : j < m, T_j^M(w) > T_{i-1}^{ex,M}(w), w(T_{j+1}^M(w)) \notin \Delta_i\},$$

if the minimum exists, otherwise $j(i) = m$. Then define $T_i^{ex,M}(w) = T_{j(i)}^M(w)$, and let Δ_{i+1} be the unique 2^M -triangle that contains both $w(T_i^{ex,M})$ and $w(T_{j(i)+1}^M)$. By definition, we see that $\Delta_i \cap \Delta_{i+1}$ is a one-point set $\{w(T_i^{ex,M})\}$, for $i = 1, \dots, k-1$. We denote the sequence of these triangles by $\sigma_M(w) = (\Delta_1, \dots, \Delta_k)$, and call it the 2^M -**skeleton** of w . We call the sequence $\{T_i^{ex,M}(w)\}_{i=0,1,\dots,k}$ **exit times** from the triangles in the skeleton. For each i , there is an $n = n(i)$ such that $T_{i-1}^{ex,M}(w) = T_n^M(w)$. We say $\Delta_i \in \sigma_M(w)$ is an element of **Type 1** if $T_i^{ex,M}(w) = T_{n+1}^M$, and an element of **Type 2** if $T_i^{ex,M}(w) = T_{n+2}^M$. If $w \in \hat{W}_N \cup \hat{V}_N$ for some N , then $\Delta_1, \dots, \Delta_k$ are mutually distinct, and each of them is either of Type 1 or of Type 2.

Assume $w \in W_N \cup V_N$ for some N and $M \leq N$. For each Δ in $\sigma_M(w)$, the **path segment of w in Δ** is

$$[w(n), T_{i-1}^{ex,M}(w) \leq n \leq T_i^{ex,M}],$$

and it is denoted by $w|_\Delta$. Note that the definition of T_i^M ’s allows a path segment $w|_\Delta$ to leak into two neighboring 2^M -triangles. It should be noted that the subgraph contained in Δ and its neighboring triangles has the same structure as $\triangle Oa_Mb_M$ and its neighbors, which implies that $w|_\Delta$ can be naturally identified with some path in $\triangle Oa_Mb_M$ and its neighbors starting at O , by translation, rotation and reflection. For convenience we shall denote this identification by η , and write:

$$\eta(w|_\Delta) = v \in W_M \cup V_M, \quad (1)$$

where the entrance to Δ is mapped to O and the exit to a_M .

To introduce the loop-erasing operation for paths in $W_N \cup V_N$, let us take a loop $[w(i), w(i+1), \dots, w(i+i_0)]$ that is contained in $w \in W_N \cup V_N$, and define its diameter by $d = \max\{i < j \leq i+i_0 : |w(j) - w(i)|\}$. The loop $[w(i), w(i+1), \dots, w(i+i_0)]$ is said to be a 2^M -**scale loop**, whenever there exists an $M \in \mathbb{Z}_+$ such that

$$\max\{N' : w(i) = w(i+i_0) \in G_{N'}\} = M \text{ and } d \geq 2^M.$$

Then the definition implies that w has a 2^{N-1} -scale loop if and only if the coarse-grained path $Q_{N-1}w$ has a loop. The operation of erasing largest-scale loops can be reduced to erasing loops from a path in $W_1 \cup V_1$, which we shall show below by induction.

Let $w \in W_N \cup V_N$ (Fig. 3(a)). we define the operation of ‘erasing the largest-scale loops’ as follows:

- 1) Coarse-grain w to obtain

$$w' = Q_{N-1}w = [w(T_0^{N-1}), w(T_1^{N-1}), \dots, w(T_k^{N-1})],$$

where $w(T_k^{N-1}) = a_N$ (Fig. 3(b)). We note that $2^{-(N-1)}w' \in W_1 \cup V_1$.

- 2) Similarly to the procedure for $W_1 \cup V_1$, erase loops from w' , using the following sequence and defining the mapping L :

$$s_0 = \sup\{j : w(T_j^{N-1}) = O\},$$

$$s_i = \sup\{j : w(T_j^{N-1}) = w(T_{s_{i-1}+1}^{N-1})\}, \quad i \geq 1,$$

and

$$Lw' = [w(T_{s_0}^{N-1}), w(T_{s_1}^{N-1}), \dots, w(T_{s_m}^{N-1})],$$

where $w(T_{s_m}^{N-1}) = a_N$ (Fig. 3(c)). We note here that $2^{-(N-1)}Lw' \in \hat{W}_1 \cup \hat{V}_1$.

- 3) Make a path by concatenation of m parts chosen from the original path ;

$$L_{N-1}w = [w_0, w_1, \dots, w_{m-1}, a_N],$$

where

$$w_i = [w(T_{s_i}^{N-1}), w(T_{s_i}^{N-1} + 1) \dots, w(T_{s_{i+1}}^{N-1} - 1)], \quad i = 0, \dots, m-1.$$

By steps 1)–3), we have obtained $L_{N-1}w \in W_N \cup V_N$ with all 2^{N-1} -scale loops of w erased (Fig. 3(d)).

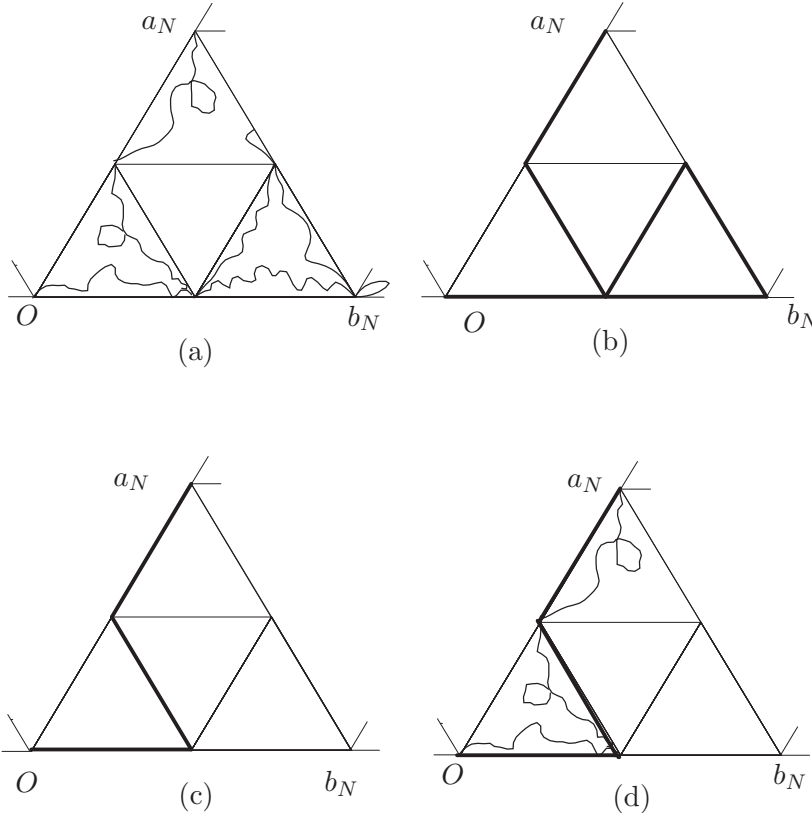


Fig. 3

Using above as a base step, we shall now describe the induction step of our operation: Let $w \in W_N \cup V_N$. For $M \leq N$, assume that all of the 2^N - to 2^M -scale loops have been erased from the path w , and denote the resulting path w' , and its 2^M -skeleton by $\sigma_M(w')$. Additionally, for each $\Delta \in \sigma_M(w')$, we shall (implicitly) use the identification η defined in (1) to identify $Q_{M-1}w'|_\Delta$ with a path in $W_1 \cup V_1$.

L1) Coarse-grain w' to obtain $Q_{M-1}w'$ and consider

$$Q_{M-1}w'|_\Delta = [w'(T_k^{M-1}), w'(T_{k+1}^{M-1}), \dots, w'(T_{k+k_0}^{M-1})],$$

where $w'(T_k^{M-1})$ is the entrance point to Δ and $w'(T_{k+k_0}^{M-1})$ the exit point from Δ .

L2) Erase loops from $Q_{M-1}w'|_\Delta$ as in the procedure for $W_1 \cup V_1$ by defining the sequence $\{s_i\}_{i=1}^m$ by

$$\begin{aligned} s_0 &= \sup\{j : w'(T_j^{M-1}) = w'(T_k^{M-1})\}, \\ s_i &= \sup\{j : w'(T_j^{M-1}) = w'(T_{s_{i-1}+1}^{M-1})\}, \quad i \geq 1, \end{aligned}$$

and denoting

$$L(Q_{M-1}w'|_\Delta) = [w'(T_{s_0}^{M-1}), w'(T_{s_1}^{M-1}), \dots, w'(T_{s_m}^{M-1})],$$

where $w'(T_{s_0}^{M-1}) = w'(T_k^{M-1})$ and $w'(T_{s_m}^{M-1}) = w'(T_{k+k_0}^{M-1})$.

L3) Make a path segment in Δ by concatenation of m parts chosen from the original path and the exit point and denote it by

$$L_{M-1}(w|_\Delta) = [w'_0, w'_1, \dots, w'_{m-1}, w'(T_{s_m}^{M-1})],$$

where

$$w'_i = [w'(T_{s_i}^{M-1}), w'(T_{s_i}^{M-1} + 1), \dots, w'(T_{s_{i+1}}^{M-1} - 1)], \quad i = 0, \dots, m-1.$$

L4) Make a whole path $w'' = L_{M-1}w$ by concatenation of parts obtained in L3) over all $\Delta \in \sigma_M(w')$.

Thus, by the procedure above, we have erased all of the 2^{M-1} -scale loops from w . Now denote by $\hat{Q}_{M-1}w$ the path obtained by concatenation of $L(Q_{M-1}w'|_\Delta)$ obtained in L2); then it is a path on F_{M-1} , in the sense that $Q_{M-1}(\hat{Q}_{M-1}w) = \hat{Q}_{M-1}w$, from O to a_N without loops. Observe that $\hat{Q}_{M-1}w = Q_{M-1}w''$. Although it may occur that $\sigma_{M-1}(w'') \neq \sigma_{M-1}(w')$, it holds that $\sigma_M(w'') = \sigma_M(w')$, which can be extended to $\sigma_K(w') = \sigma_K(w'')$ for any $K \geq M$.

We then continue this operation until we have erased all of the loops and have $Lw = L_0w = \hat{Q}_0w$. Thus, by construction, our loop-erasing operation is essentially a repetition of loop-erasing for $W_1 \cup V_1$. We remark that the procedure implies that for any $w \in W_N \cup V_N$,

$$\sigma_K(\hat{Q}_M w) = \sigma_K(\hat{Q}_K w) \quad \text{for any } M \leq K \leq N. \quad (2)$$

i.e., in the process of loop-erasing, once loops of 2^K -scale and greater have been erased, the 2^K -skeleton does not change any more. However it should be noted that the types of the triangles can change from Type 2 to Type 1.

2.4 Loop-erased random walks on the pre-Sierpiński gaskets.

Let $(\tilde{\Omega}, \mathcal{F}, P)$ be a probability space. A simple random walk on F_0 is a G_0 -valued Markov chain $\{Z(i) : i \in \mathbb{Z}_+\}$ with transition probabilities

$$P[Z(i+1) = y \mid Z(i) = x] = \begin{cases} \frac{1}{4} & \text{if } y \in \mathcal{N}_0(x) \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this paper, we will consider random walks starting at O , so finite random walk paths are elements of W^* , and thus, T_i^N 's and $Q_N Z$ can be defined.

Consider two kinds of random walks stopped at a_N : one conditioned on $Z(T_1^N) = a_N$ (before hitting other G_N vertices), called X_N , and the other conditioned on $Z(T_1^N) = b_N$ and $Z(T_2^N) = a_N$, i.e. hitting b_N on the way to a_N , called X'_N . These random walks then induce measures P_N and P'_N on W^* with support on W_N and V_N , respectively, namely,

$$\begin{aligned} P_N[w] &= P[X_N(i) = w(i), i = 0, 1, \dots, \ell(w)] \\ &= P[Z(i) = w(i), i = 0, 1, \dots, \ell(w) \mid Z(T_1^N) = a_N], \quad w \in W_N, \end{aligned}$$

$$\begin{aligned} P'_N[w] &= P[X'_N(i) = w(i), i = 0, 1, \dots, \ell(w)] \\ &= P[Z(i) = w(i), i = 0, 1, \dots, \ell(w) \mid Z(T_1^N) = b_N, Z(T_2^N) = a_N], \quad w \in V_N. \end{aligned}$$

Note that by symmetry:

$$P[Z(T_1^N) = a_N] = 1/4, \quad P[Z(T_1^N) = b_N, Z(T_2^N) = a_N] = 1/16.$$

Throughout this paper, the following propositions on the simple random walks on the pre-Sierpiński gasket will be used; They are straightforward consequences of the ‘self-similarity’, that is, $2^{-M}F_M = F_0$, and the property that if $x_0 \in G_M$ for some $M \in \mathbb{Z}_+$, then for each $x \in \mathcal{N}_M(x_0)$

$$P[Z(T_{i+1}^M) = x \mid Z(T_i^M) = x_0] = \frac{1}{4}$$

holds. (For details of random walks on the Sierpiński gasket, we refer to [2].)

Proposition 1 *If $M \leq N$, then the distributions of $2^{-M}Q_M X_N$ and $2^{-M}Q_M X'_N$ are equal to P_{N-M} and P'_{N-M} , respectively; in other words, $Q_M X_N$ and $Q_M X'_N$ are simple random walks on a coarse graph F_M stopped at a_N .*

Let η be the identification map defined in the last subsection.

Proposition 2 *Let $M \leq N$, and consider random walk segments conditioned on $Q_M X_N$ between the hitting times,*

$$Z_i = [X_N(t), T_i^M(X_N) \leq t \leq T_{i+1}^M(X_N)], \quad i = 1, \dots, m,$$

where $X_N(T_m^M) = a_N$. Then $Z_i, i = 1, \dots, m$, when identified with paths in W_{N-M} by appropriate translation, rotation and reflection, are independent and have the same distribution as X_{N-M} .

By applying loop-erasing operation to random walks X_N and X'_N , we induce measures $\hat{P}_N = P_N \circ L^{-1}$ supported on \hat{W}_N , and $\hat{P}'_N = P'_N \circ L^{-1}$ supported on $\hat{W}_N \cup \hat{V}_N$, respectively. Paths in \hat{W}_1 and \hat{V}_1 are shown in Fig. 4.

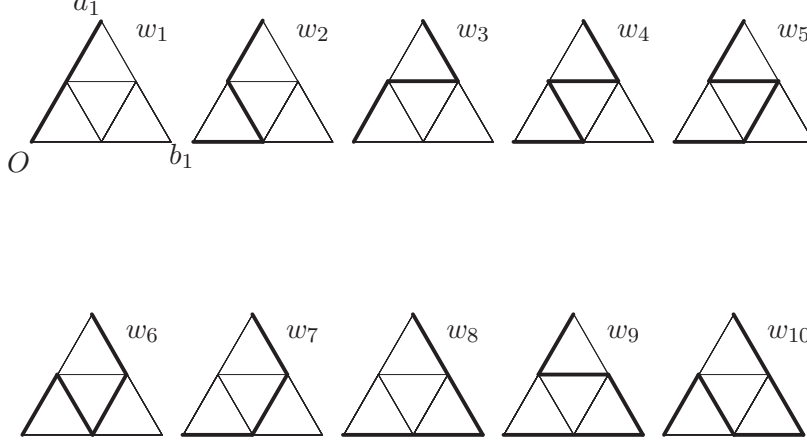


Fig. 4

Their probabilities under \hat{P}_1 and \hat{P}'_1 , respectively, can be obtained by direct calculation:

$$\begin{aligned} \hat{P}_1[w_1] &= \frac{1}{2}, \hat{P}_1[w_2] = \frac{2}{15}, \hat{P}_1[w_3] = \frac{2}{15}, \hat{P}_1[w_4] = \frac{1}{30}, \hat{P}_1[w_5] = \frac{1}{30}, \hat{P}_1[w_6] = \frac{1}{30}, \hat{P}_1[w_7] = \frac{2}{15}, \\ \hat{P}'_1[w_1] &= \frac{1}{9}, \hat{P}'_1[w_2] = \frac{11}{90}, \hat{P}'_1[w_3] = \frac{11}{90}, \hat{P}'_1[w_4] = \frac{2}{45}, \hat{P}'_1[w_5] = \frac{2}{45}, \hat{P}'_1[w_6] = \frac{2}{45}, \\ \hat{P}'_1[w_7] &= \frac{8}{45}, \hat{P}'_1[w_8] = \frac{2}{9}, \hat{P}'_1[w_9] = \frac{1}{18}, \hat{P}'_1[w_{10}] = \frac{1}{18}. \end{aligned}$$

For $w \in \hat{W}_N \cup \hat{V}_N$, let us denote the number of Type 1 triangles and Type 2 triangles in $\sigma_0(w)$ by $s_1(w)$ and $s_2(w)$, respectively. (This implies that $\ell(w) = s_1(w) + 2s_2(w)$.) Define two sequences, $\{\Phi_N\}_{N \in \mathbb{N}}$ and $\{\Theta_N\}_{N \in \mathbb{N}}$, of generating functions by:

$$\begin{aligned} \Phi_N(x, y) &= \sum_{w \in \hat{W}_N} \hat{P}_N(w) x^{s_1(w)} y^{s_2(w)}, \\ \Theta_N(x, y) &= \sum_{w \in \hat{V}_N} \hat{P}'_N(w) x^{s_1(w)} y^{s_2(w)}, \quad x, y \geq 0. \end{aligned}$$

For simplicity, we shall denote $\Phi_1(x, y)$ and $\Theta_1(x, y)$ by $\Phi(x, y)$ and $\Theta(x, y)$.

Proposition 3 *The above generating functions satisfy the following recursion relations for all $N \in \mathbb{N}$:*

$$\begin{aligned} \Phi(x, y) &= \frac{1}{30}(15x^2 + 8xy + y^2 + 2x^2y + 4x^3). \\ \Theta(x, y) &= \frac{1}{45}(5x^2 + 11xy + 2y^2 + 14x^2y + 8x^3 + 5xy^2). \\ \Phi_{N+1}(x, y) &= \Phi_N(\Phi(x, y), \Theta(x, y)). \\ \Theta_{N+1}(x, y) &= \Theta_N(\Phi(x, y), \Theta(x, y)). \end{aligned}$$

Proof. We shall first express \hat{P}_{N+1} in terms of \hat{P}_N , \hat{P}_1 and \hat{P}'_1 . If we recall the procedure for obtaining $\hat{Q}_1 X_{N+1}$ from X_{N+1} , we notice that it is the same as the procedure to obtain LX_N from X_N , except that everything is twice larger in the case of X_{N+1} . This together with Proposition 1 implies that the distribution of $2^{-1}\hat{Q}_1 X_{N+1}$ is equal to \hat{P}_N , namely,

$$P_{N+1}[v : \frac{1}{2}\hat{Q}_1 v = u] = \hat{P}_N[u].$$

On the other hand, we have from (2)

$$\sigma_1(\hat{Q}_1 X_{N+1}) = \sigma_1(LX_{N+1}).$$

The rest of the loop-erasing procedure to obtain LX_{N+1} together with Proposition 2 implies that conditioned on $\hat{Q}_1 X_{N+1}$, the walk segments of $L_1 X_{N+1}$ in $\Delta \in \sigma_1(\hat{Q}_1 X_{N+1})$ have the same distribution as either X_1 or X'_1 (modulo appropriate transformation), and that they are mutually independent, which further implies that $LX_{N+1}|\Delta$ are independent.

Keeping these observations in mind, we calculate $\hat{P}_{N+1}[w]$ for $w \in \hat{W}_{N+1}$. Let $\sigma_1(w) = (\Delta_1, \dots, \Delta_k)$ be the 2^1 -skeleton of w and let $w_i = w|_{\Delta_i}$ and let ηw_i be their identification with paths in $W_1 \cup V_1$ as defined in (1). Let \sum_u denote the sum taken over $u \in \hat{W}_N$ satisfying $\sigma_0(u) = \frac{1}{2}\sigma_1(w)$, which consists of $\Delta_1, \dots, \Delta_k$ scaled by $1/2$.

Thus, we have

$$\begin{aligned} \hat{P}_{N+1}[w] &= P_{N+1}[v : Lv = w] \\ &= \sum_u P_{N+1}[Lv = w, \frac{1}{2}\hat{Q}_1 v = u] \\ &= \sum_u P_{N+1}[Lv = w | \frac{1}{2}\hat{Q}_1 v = u] P_{N+1}[\frac{1}{2}\hat{Q}_1 v = u] \\ &= \sum_u P_{N+1}[Lv = w | \frac{1}{2}\hat{Q}_1 v = u] \hat{P}_N[u] \\ &= \sum_u P_{N+1}[\eta(Lv|_{\Delta_i}) = \eta w_i, i = 1, \dots, k | \frac{1}{2}\hat{Q}_1 v = u] \hat{P}_N[u] \\ &= \sum_u (\prod_{i=1}^k \hat{P}_1^*[\eta w_i]) \hat{P}_N[u], \end{aligned}$$

where $\hat{P}_1^* = \hat{P}_1$ if Δ_i is of Type 1, and $\hat{P}_1^* = \hat{P}'_1$ if Δ_i is of Type 2.

Since taking the sum over $w \in \hat{W}_{N+1}$ means taking the sum over all $u \in \hat{W}_N$ and finer structures in each $\Delta \in \sigma_1(w)$, we have

$$\begin{aligned} \Phi_{N+1}(x, y) &= \sum_{w \in \hat{W}_{N+1}} \hat{P}_{N+1}(w) x^{s_1(w)} y^{s_2(w)} \\ &= \sum_{u \in \hat{W}_N} \sum_{\eta w_1 \in \hat{W}_1^*} \cdots \sum_{\eta w_k \in \hat{W}_1^*} (\prod_{i=1}^k \hat{P}_1^*[\eta w_i]) \hat{P}_N[u] x^{s_1(w_1) + \cdots + s_1(w_k)} y^{s_2(w_1) + \cdots + s_2(w_k)} \\ &= \sum_{u \in \hat{W}_N} \hat{P}_N[u] \prod_{i=1}^k (\sum_{w_i \in \hat{W}_1^*} \hat{P}_1^*[w_i] x^{s_1(w_i)} y^{s_2(w_i)}) \\ &= \sum_{u \in \hat{W}_N} \hat{P}_N[u] \Phi(x, y)^{s_1(u)} \Theta(x, y)^{s_2(u)} \\ &= \Phi_N(\Phi(x, y), \Theta(x, y)). \end{aligned}$$

The calculations for \hat{P}'_{N+1} and $\Theta_{N+1}(x, y)$ are similar. \square

Define the mean matrix by

$$\mathbf{M} = \begin{bmatrix} \frac{\partial}{\partial x}\Phi(1, 1) & \frac{\partial}{\partial y}\Phi(1, 1) \\ \frac{\partial}{\partial x}\Theta(1, 1) & \frac{\partial}{\partial y}\Theta(1, 1) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & \frac{2}{15} \\ \frac{26}{15} & \frac{13}{15} \end{bmatrix}. \quad (3)$$

It is a strictly positive matrix, and the larger eigenvalue is

$$\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\dots$$

The loop-erasing procedure together with Proposition 2 leads to

Proposition 4 *Let $M \leq N$. Conditioned on $\sigma_M(LX_N) = (\Delta_1, \dots, \Delta_k)$ and the types of each element of the skeleton, the traverse times of the triangles*

$$T_i^{ex, M}(LX_N) - T_{i-1}^{ex, M}(LX_N), \quad i = 1, 2, \dots, k$$

are independent. Each of them has the same distribution as either $T_1^{ex, N-M}(LX_{N-M})$ or $T_1^{ex, N-M}(LX'_{N-M})$, according to whether Δ_i is of Type 1 or Type 2.

Theorem 5 *As $N \rightarrow \infty$, $\lambda^{-N}\ell(LX_N)$ converges in law to an integrable random variable W' , with a positive probability density.*

We shall prove the above theorem in Section 3, using coupling argument. Theorem 5 suggests that the displacement exponent for the loop-erased random walk on the pre-Sierpiński gasket is $\log \lambda / \log 2$, in the sense that the average number of steps it takes to cover the distance of 2^N is of order λ^N . In other words, if we write $m = 2^N$, it takes $m^{\log \lambda / \log 2}$ steps to travel a distance of m from the origin. This value is equal to that obtained by Shinoda [16] who defined a loop-erased walk through uniform spanning trees.

3 Scaling limit of the loop-erased random walks.

3.1 Paths on the Sierpiński gasket.

In this section we investigate the limit of the loop-erased random walk as the lattice spacing (edge length) tends to 0. First we define the (finite) Sierpiński gasket. Since it will be easier to deal with continuous functions from the beginning, we regard F_0 as a closed subset of \mathbb{R}^2 made up of all the points on its edges. Let Δ_1 be the closed (filled) triangle in \mathcal{T}_0 whose vertices are O, a_0 and b_0 , and Δ_2 be its reflection with regard to the y -axis, and let $F^N = 2^{-N}F_0 \cap (\Delta_1 \cup \Delta_2)$ (Fig 5). We define the **Sierpiński gasket** by $F = cl(\cup_{N=0}^{\infty} F^N)$, where cl denotes closure. We define the sets of vertices by $G^N = 2^{-N}G_0 \cap (\Delta_1 \cup \Delta_2)$.

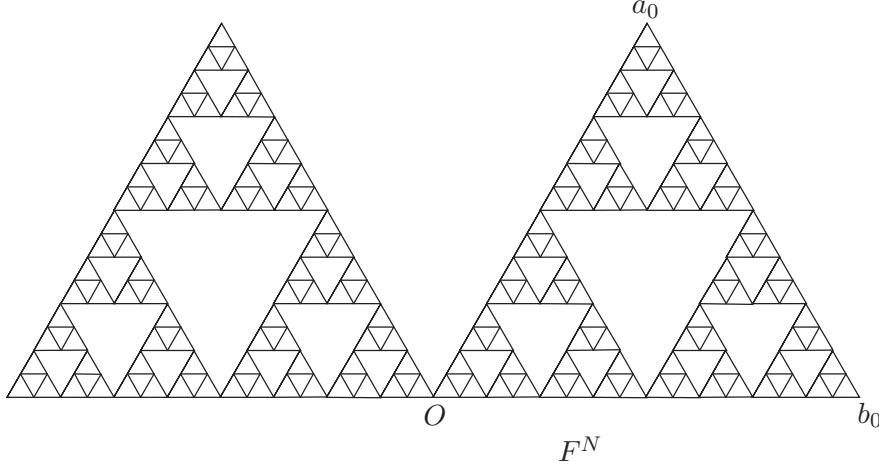


Fig. 5

Let

$$C = \{w \in C([0, \infty) \rightarrow F) : w(0) = O, \lim_{t \rightarrow \infty} w(t) = a_0\}.$$

C is a complete separable metric space with the metric

$$d(u, v) = \sup_{t \in [0, \infty)} |u(t) - v(t)|, \quad u, v \in C,$$

where $|x - y|$, $x, y \in \mathbb{R}^2$, denotes the Euclidean distance. Throughout this section, for $w \in \bigcup_{N=1}^{\infty} W_N$, we let

$$w(t) = a_N, \quad t \geq \ell(w),$$

and interpolate all the paths linearly,

$$w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), \quad i \leq t < i + 1, \quad i = 0, 1, 2, \dots$$

so that we can regard w as a continuous function on $[0, \infty)$.

Let

$$W^N = 2^{-N}W_N = \{2^{-N}w : w \in W_N\}, \quad \hat{W}^N = 2^{-N}\hat{W}_N,$$

where all the paths in W^N are understood to have been linearly interpolated. In the following we shall use this identification modulo linear interpolation. Thus, W^N and \hat{W}^N are subsets of C . For $w \in W^N$, let $\tilde{\ell}(w) = \ell(2^N w)$. Namely, $\tilde{\ell}(w)$ is the number of 2^{-N} -sized ‘steps’ the path w takes to get to a_0 .

We define hitting times, coarse-graining, exit times and skeletons similarly to Section 2, but with G_M replaced by G^M . Namely, for $w \in C$ we define a sequence $\{T_i^M(w)\}_{i=0}^m$ of the hitting times of G^M , as follows: $T_0^M(w) = 0$, and for $i \geq 1$, let $T_i^M(w) = \inf\{j > T_{i-1}^M(w) : w(j) \in G^M \setminus \{w(T_{i-1}^M(w))\}\}$. m is the smallest integer such that $T_{m+1}^M(w) = \infty$. For the hitting times we are using the same notation but we hope no confusion arises. For $N \in \mathbb{Z}_+$, we define a coarse-graining map $Q^N : C \rightarrow C$ by $(Q^N w)(i) = w(T_i^N(w))$ for $i = 0, 1, 2, \dots, m$, and by using linear interpolation

$$(Q^N w)(t) = \begin{cases} (i + 1 - t) (Q^N w)(i) + (t - i) (Q^N w)(i + 1), & i \leq t < i + 1, \quad i = 0, 1, 2, \dots, m - 1, \\ a_0, & t \geq m. \end{cases}$$

Notice that

$$Q^M \circ Q^N = Q^M, \quad \text{if } M \leq N \quad (4)$$

holds.

Since we have defined the hitting times for every $w \in C$, we can define its 2^{-M} -**skeleton**, $\sigma^M(w)$ (a sequence of 2^{-M} -triangles w passes through) and the **exit times** $\{T_i^{ex,M}\}$ similarly to their counterparts in Section 2. To define the loop erasing operator, recall that if $w \in W^N$, then $2^N w \in W_N$ and $L(2^N w) \in \hat{W}_N$ (modulo linear interpolation). Thus we define loop erasure $\tilde{L} : \bigcup_{N=0}^{\infty} W^N \rightarrow \bigcup_{N=0}^{\infty} \hat{W}^N$ by letting $\tilde{L}w = 2^{-N} L(2^N w) \in \hat{W}^N$ for $w \in W^N$, $N \in \mathbb{Z}_+$, and we define also $\hat{Q}^M w = 2^{-N} \hat{Q}_M(2^N w) \in \hat{W}^M$ for $M \leq N$. The only differences from the previous section are that paths are continuous (by linear interpolation) and confined in two neighboring unit triangles, and that we erase loops from 2^{-1} -scale down. For each $N \in \mathbb{Z}_+$, let P^N be the random walk path measure on F^N (a probability measure on C supported on W^N), namely $P^N[w] = P_N[2^N w]$, for $w \in W^N$. In the following, we will focus on P^N . V_N 's and P'_N 's introduced in the previous section have played auxiliary roles.

3.2 The scaling limit.

We consider random walks (linearly interpolated version) on G^N , $N \in \mathbb{Z}_+$, starting at O and stopped at a_0 .

Let

$$\Omega' = \{\omega = (w_0, w_1, w_2, \dots) : w_0 \in \hat{W}^0, w_N \in \hat{W}^N, w_N \triangleright w_{N+1}, N \in \mathbb{N}\},$$

where $w_N \triangleright w_{N+1}$ means that there exists a $v \in W^{N+1}$ such that $Q^N v = w_N$ and $\hat{Q}^{N+1} v = w_{N+1}$. Define the projection onto the first $N+1$ elements by

$$\pi_N \omega = (w_0, w_1, \dots, w_N),$$

and a probability measure on $\pi_N \Omega'$ by

$$\hat{P}^N[(w_0, w_1, \dots, w_N)] = P^N[v : \hat{Q}^i v = w_i, i = 0, \dots, N]$$

The following consistency condition is a direct consequence of the loop-erasing procedure:

$$\hat{P}^N[(w_0, w_1, \dots, w_N)] = \sum_u \hat{P}^{N+1}[(w_0, w_1, \dots, w_N, u)], \quad (5)$$

where the sum is taken over all possible u such that $w_N \triangleright u$.

By virtue of (5) and Kolmogorov's extension theorem for a projective limit, there is a probability measure \hat{P} on $\Omega_0 = C^{\mathbb{N}} = C \times C \times \dots$ such that

$$\hat{P}[\Omega'] = 1.$$

$$\hat{P} \circ \pi_N^{-1} = \hat{P}^N, \quad N \in \mathbb{Z}_+.$$

Let $Y^N : \Omega' \rightarrow C$ be the projection to the N -th component. We regard Y^N as an F -valued process $Y^N(\omega, t)$ on $(\Omega_0, \mathcal{B}, \hat{P})$, where \mathcal{B} is the Borel algebra on Ω_0 generated by the cylinder sets.

For $w \in C$ and $j = 1, 2$, denote by $S_j^M(w)$ the number of 2^{-M} -triangles of Type j in $\sigma^M(w)$, namely, $S_j^M(w) = \#\{i : \Delta_i \text{ is of Type } j\}$, and let $\mathbf{S}^M(w) = (S_1^M(w), S_2^M(w))$. If $w \in W^N$ for some N , then $\hat{\ell}(w) = S_1^N(w) + 2S_2^N(w)$.

Let $\mathbf{S} = (S_1, S_2)$ and $\mathbf{S}' = (S'_1, S'_2)$ be \mathbb{Z}_+ -valued random variables on $(\Omega_0, \mathcal{B}, \hat{P})$ with the same distributions as those of (s_1, s_2) under \hat{P}_1 and under \hat{P}'_1 , respectively. (s_1, s_2) has been defined in 2.4 together with the generating functions.

Proposition 6 Fix arbitrarily $v \in \hat{W}^M$, and let $\sigma^M(v) = (\Delta_1, \dots, \Delta_k)$. For each i , $1 \leq i \leq k$, under the conditional probability $\hat{P}[\cdot | Y^M = v]$, $\{\mathbf{S}^{M+N}(Y^{M+N} | \Delta_i), N = 0, 1, 2, \dots\}$ is a two-type supercritical branching process, with the types of children corresponding to the types of triangles. The offspring distributions born from a Type 1 triangle and from a Type 2 triangle are equal to those of \mathbf{S} and \mathbf{S}' , respectively. If Δ_i is of Type 1, the process initiates in state $(1, 0)$, and if Δ_i is of Type 2, in state $(0, 1)$.

(1) The generating functions for the offspring distributions are

$$g_1(x, y) \stackrel{\text{def}}{=} \hat{E}[x^{S_1} y^{S_2}] = \Phi(x, y),$$

$$g_2(x, y) \stackrel{\text{def}}{=} \hat{E}[x^{S'_1} y^{S'_2}] = \Theta(x, y),$$

where \hat{E} is an expectation with regard to \hat{P} .

(2) The mean matrix \mathbf{M} is given by (3) in Section 2. It is strictly positive and its eigenvalues are $\lambda = \frac{1}{15}(20 + \sqrt{205}) = 2.2878\dots$ and $\lambda' = \frac{1}{15}(20 - \sqrt{205}) = 0.3788\dots$. We have

$$\hat{E}[\mathbf{S}^{M+N}(Y^{M+N} | \Delta_i) | Y^M = v] = \mathbf{S}^M(v | \Delta_i) \mathbf{M}^N.$$

(3) $\hat{P}[S_1 + S_2 \geq 2] = \hat{P}[S'_1 + S'_2 \geq 2] = 1$ (non-singularity).

(4)

$$\hat{E}[S_i \log S_i] < \infty, \quad \hat{E}[S'_i \log S'_i] < \infty, \quad i = 1, 2.$$

Proposition 6 suggests that we should consider F -valued processes with time appropriately scaled. Thus, we introduce a time-scale transformation $U_N(\alpha) : C \rightarrow C$, $\alpha \in (0, \infty)$, $n \in \mathbb{N}$. For $w \in C$, define

$$(U_N(\alpha)w)(t) \stackrel{\text{def}}{=} w(\alpha^N t),$$

and consider the processes

$$X^N = U_N(\lambda)Y^N, \quad N \in \mathbb{Z}_+.$$

Proposition 7

$$\sigma^M(X^N) = \sigma^M(X^M) = \sigma^M(Y^M), \quad M \leq N, \quad a.s.$$

In particular,

$$X^N(T_i^{ex, M}(X^N)) = X^M(T_i^{ex, M}(X^M)) = Y^M(T_i^{ex, M}(Y^M)), \quad M \leq N, \quad a.s. \quad (6)$$

Note that if $\sigma^M(X^N) = (\Delta_1, \dots, \Delta_k)$, then

$$T_j^{ex, M}(X^N) = \lambda^{-N} \sum_{i=1}^j (S_1^N(X^N | \Delta_i) + 2S_2^N(X^N | \Delta_i)), \quad 1 \leq j \leq k.$$

Proposition 6 combined with the convergence theorem for supercritical branching processes (see [1], Chapter V) leads to the following proposition.

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be the right and left positive eigenvectors associated with λ such that $|\mathbf{u}| = |\mathbf{v}| = 1$.

Proposition 8 Fix arbitrarily $v \in \hat{W}^M$, and let $\sigma^M(v) = (\Delta_1, \dots, \Delta_k)$. For each i , $1 \leq i \leq k$, under the conditional probability $\hat{P}[\cdot | Y^M = v]$, we have the following.

- (1) For each $i \in \{1, \dots, k\}$, $\{\lambda^{-(M+N)} \mathbf{S}^{M+N}(X^{M+N} | \Delta_i), N = 0, 1, 2, \dots\}$ converges a.s. as $N \rightarrow \infty$ to a \mathbb{R}^2 -valued random variable $\mathbf{S}^{*M,i} = (S_1^{*M,i}, S_2^{*M,i})$.
- (2) $\{\mathbf{S}^{*M,i}, i = 1, \dots, k\}$ are independent.
- (3) There are random variables B_1 and B_2 such that $\mathbf{S}^{*M,i}$ is equal in distribution to $\lambda^{-M} B_1 \mathbf{v}$ if Δ_i is of Type 1, and equal in distribution to $\lambda^{-M} B_2 \mathbf{v}$ if Δ_i is of Type 2.
- (4)

$$\hat{P}[B_i > 0] = 1, \quad \hat{E}[B_i] = u_i, \quad i = 1, 2.$$

B_1 and B_2 have strictly positive probability density functions.

- (5) The Laplace transform of B_i , $i = 1, 2$

$$\phi_i(t) = \hat{E}[\exp(tB_i)]$$

are entire functions on \mathbb{C} and are the unique solution to

$$\phi_1(\lambda t) = \Phi(\phi_1(t), \phi_2(t)), \quad \phi_2(\lambda t) = \Theta(\phi_1(t), \phi_2(t)), \quad \phi_1(0) = \phi_2(0) = 1.$$

To be precise, (1)–(4) in Proposition 8 are the straightforward consequences of general limit theorems for superbranching processes (Theorem 1 and Theorem 2 in V.6 of [1]). $\hat{P}[B_i > 0] = 1$ is a consequence of Φ and Θ having no terms with degree smaller than 2. For the existence of the Laplace transform on the entire \mathbb{C} , we need careful study of the recursions. We omit the details here, since they are similar to those in [9].

Let $T_i^{*M} = \sum_{j=1}^i (S_1^{*M,j} + 2S_2^{*M,j})$. Then $\lim_{N \rightarrow \infty} T_j^{ex,M}(X^N) = T_j^{*M}$. By virtue of Proposition 7 and Proposition 8, we can prove the almost sure uniform convergence for X^N .

Theorem 9 X^N converges uniformly in t a.s. as $N \rightarrow \infty$ to a continuous process X .

Proof. Choose $\omega \in \Omega'$ such that for all $M \in \mathbb{Z}_+$ the following holds: $Y^M \in \hat{W}^M$, $\lim_{N \rightarrow \infty} T_i^{ex,M}(X^N) = T_i^{*M}$ exists and $T_i^{*M} - T_{i-1}^{*M} > 0$ for all $1 \leq i \leq k_M$, where k_M denotes the number of triangles in $\sigma^M(Y^M)$. Let $R = T_1^{*0} + \varepsilon$, where $\varepsilon > 0$ is arbitrary. It suffices to show that $X^N(\omega, t)$ converges uniformly in $t \in [0, R]$. In fact, if $t > R$, $X^N(t) = a_0$ for a large enough N .

Fix $M \geq 0$. Let $k = k_M$. By expressing the arrival time at a_0 as the sum of traversing times of 2^{-M} -triangles, we have $T_k^{ex,M}(X^N) = T_1^{ex,0}(X^N)$ a.s. Letting $N \rightarrow \infty$, we have $T_k^{*M} = T_1^{*0}$ a.s.

The choice of ω shows that there exists an $N_1 = N_1(\omega) \in \mathbb{N}$ such that

$$\max_{1 \leq i \leq k} |T_i^{ex,M}(X^N) - T_i^{*M}| \leq \min_{1 \leq i \leq k} (T_i^{*M} - T_{i-1}^{*M}), \quad (7)$$

and

$$|T_k^{ex,M}(X^N) - T_k^{*M}| < \varepsilon,$$

for $N \geq N_1$.

If $0 \leq t < T_k^{*M}$, then choose $j \in \{1, \dots, k\}$ such that $T_{j-1}^{*M} \leq t < T_j^{*M}$.

Then (7) implies that $T_{j-2}^{ex,M}(X^N) \leq t \leq T_{j+1}^{ex,M}(X^N)$, for $N \geq N_1$. Since Proposition 7 shows

$$X^N(T_j^{ex,M}(X^N)) = X^M(T_j^{ex,M}(X^M)), \quad (8)$$

for all N with $N \geq M$, we have

$$|X^N(T_j^{ex,M}(X^N)) - X^N(t)| \leq 3 \cdot 2^{-M}.$$

Otherwise, if $T_k^{*M} \leq t \leq T_k^{*M} + \varepsilon = R$, then let $j = k$. Since $T_{k-1}^{ex,M}(X^N) \leq t$,

$$|X^N(T_j^{ex,M}(X^N)) - X^N(t)| \leq 2 \cdot 2^{-M}.$$

Therefore, if $N, N' \geq N_1$, then for any $t \in [0, R]$,

$$\begin{aligned} & |X^N(t) - X^{N'}(t)| \\ & \leq |X^N(T_j^{ex,M}(X^N)) - X^N(t)| + |X^{N'}(T_j^{ex,M}(X^{N'})) - X^{N'}(t)| \\ & \quad + |X^N(T_j^{ex,M}(X^N)) - X^{N'}(T_j^{ex,M}(X^{N'}))| \\ & \leq 6 \cdot 2^{-M}, \end{aligned}$$

where the third term in the middle part is shown to be 0 by (8). Since M is arbitrary, we have the uniform convergence. \square

Theorem 10 *X is almost surely self-avoiding. The Hausdorff dimension of the path $X([0, \infty))$ is almost surely equal to $\log \lambda / \log 2$.*

The uniform convergence of X^N , which is self-avoiding, to X implies that the probability of the event that there exist t_1, t_2 and t_3 with $t_1 < t_2 < t_3$ such that $X(t_1) = X(t_3)$, $X(t_2) \neq X(t_1)$ is zero, and the existence of the Laplace transforms $\hat{E}[\exp(t_0 B_i)]$, $i = 1, 2$ for some $t_0 > 0$ guarantees that the probability that there exist $t_1, t_2 > 0$ such that $X(t) = X(t_1)$ for all t , $t_1 \leq t \leq t_1 + t_2$ is zero. We omit the proof here since they are similar to that in [9]. To calculate the Hausdorff dimension, we regard the path as a multi-type random fractal. The proof is similar to that in [10].

Since $\lambda^{-N} \ell(LX_N)$ in Theorem 5 has the same distribution as $\lambda^{-N}(S_1^N(X^N) + 2S_2^N(X^N))$, Theorem 5 follows immediately from Proposition 8, with W' equal in distribution to $(v_1 + 2v_2)B_1$.

4 Conclusion

We proposed a model of loop-erased random walks on the finite pre-Sierpiński gasket. Our loop-erasing procedure is based on a ‘larger-scale-loops-first’ rule, which enables us to obtain exact recursion relations. First, we proved the existence of the scaling limit. Then, we made use of the tools that have been developed for the study of self-avoiding walks on the pre-Sierpiński gasket to prove that the path of the limiting process is almost surely self-avoiding, while having Hausdorff dimension strictly greater than 1. Our path Hausdorff dimension is consistent with the results in [3] and [16], thus we conjecture that our model is in the same universality class as theirs.

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